Can magnetic monopoles and massive photons coexist in the framework of the same classical theory?

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It is well known that one cannot construct a self-consistent quantum field theory describing the non-relativistic electromagnetic interaction mediated by massive photons between a point-like electric charge and a magnetic monopole. We show that, indeed, this inconsistency arises in the classical theory itself. No semi-classic approximation or limiting procedure for $\hbar \to 0$ is used. As a result, the string attached to the monopole emerges as visible also if finite-range electromagnetic interactions are considered in classical framework.

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In his classical works, Dirac showed that the existence of a magnetic monopole would explain the electric charge quantization [1]. This is known as the Dirac quantization rule. There exist various arguments based on quantum mechanics, theory of representations, topology and differential geometry on behalf of Dirac's rule [2]. Dirac's formulation of magnetic monopoles takes into account a singular vector potential. Other approaches exist where two non-singular vector potentials, related through a gauge transformation, are used [3, 4]. Finite-range electrodynamics is a theory with non-zero photon mass. It is an extension of the standard theory and is fully compatible with experiments. The existence of Dirac's monopole in massless electrodynamics is compatible with the above quantization condition if the string attached to the monopole is invisible. The quantization condition can be obtained either with the help of gauge invariance or angular momentum quantization. In massive electrodynamics, both these approaches are no longer applicable. These conclusions are formulated in a quantum framework which is a quantized version of the classical one. The problem of considering a satisfactory classical relativistic framework for massive electrodynamics and magnetic monopoles has been considered in [5, 6]. In this letter, we investigate the problem at a non-relativistic classical level before taking into account any quantization procedure of the theory itself. An important distinction between classical and quantum theory is the following: in a classical theory, with arbitrary force law, there is no reason to expect a conserved total angular momentum, even if energy and linear momentum are conserved. In a quantum theory, general invariance requirements, combined with the linearity of the theory, guarantee the existence of an angular momentum vector $\vec{J}$ which commutes with the S-matrix. As a result, some classical theories might have no quantum analogue, and others might have a quantum analogue only for restricted classes of parameters, in order to allow the existence of a conserved quantized angular momentum. The latter case comes out in the framework of the classical non-relativistic electromagnetic interaction mediated by massless photons between an electric charge and a magnetic monopole [7]. The usual ground for conservation of angular momentum in classical theory is the existence of a rotationally invariant Hamiltonian. Such a Hamiltonian does exist in the case of massless photons and the quantization of the angular momentum appearing in the Hamiltonian leads to the above Dirac condition. In this letter, it is shown that this inconsistency arises in the classical theory itself. It is known that no spherically symmetric magnetic field solutions are allowed in Maxwell's classical electrodynamics with massive photons and magnetic monopoles [8]. We implement the permitted solutions in the classical non-relativistic Hamiltonian formulation, describing the finite range electromagnetic interaction between a point-like electric charge and a fixed Dirac monopole. Assuming that our theory is endowed with a well-defined canonical Poisson bracket structure, we show that the total angular momentum is the generator of rotations if the proper Poisson brackets are provided. At this point, we require proper transformation rules under spatial rotations for the allowed magnetic vector field solutions. By the additional assumption of a well-defined Poisson bracket structure among the total angular momentum, the vector position and the generalized vector momentum, we show that, indeed, only spherically symmetric magnetic fields satisfy our request. The conclusion is that any quantization procedure applied to this classical theory leads to an inconsistent quantum counterpart.

Let us start by recalling Maxwell's generalized equations in presence of massive photons and magnetic monopoles in the vector algebra formalism. In cgs units, we have

$$\vec{\nabla}\cdot\vec{E}=4\pi\rho_e-m_\gamma^2A_0,\ \vec{\nabla}\times\vec{E}=-c^{-1}\partial_t\vec{B}-4\pi c^{-1}\vec{j}_m \tag{1}$$

$$\vec{\nabla}\cdot\vec{B}=4\pi\rho_m,\ \vec{\nabla}\times\vec{B}=4\pi c^{-1}\vec{j}_e+c^{-1}\partial_t\vec{E}-m_\gamma^2\vec{A} \tag{2}$$

where $m_\gamma=\frac{\omega}{c}$ and ω is the frequency of the photon. In absence of electric fields, charges and currents, as well as the absence of magnetic current, the static monopole-like solution of this system is,

$$\vec{B}=\vec{B}^{(Dirac)}+\vec{B}_\gamma \tag{3}$$

where $\vec{B}^{(Dirac)}$ is the standard Dirac magnetic field,

$$\vec{B}^{(Dirac)}=\frac{e_m}{r^2}\hat{r} \tag{4}$$

whose divergence and curl are given by,

$$\vec{\nabla}\cdot\vec{B}^{(Dirac)}=\ 4\pi e_m\delta^{(3)}(\vec{r})\ \text{ and }\ \vec{\nabla}\times\vec{B}^{(Dirac)}=0. \tag{5}$$

The diffuse magnetic field $\vec{B}_\gamma(\vec{r})$ is given by the following general expression,

$$\vec{B}_\gamma(\vec{r})=b_\gamma^{(1)}(r,\hat{n}\cdot\vec{r})\vec{r}+b_\gamma^{(2)}(r,\hat{n}\cdot\vec{r})\hat{n} \tag{6}$$

where $b_\gamma^{(1)}$ and $b_\gamma^{(2)}$ are general scalar field functions and $\hat{n}$ is a unitary vector along the monopole string. The magnetic field $\vec{B}_\gamma(\vec{r})$ is such that,

$$\vec{\nabla}\cdot\vec{B}_\gamma=0\quad\text{and, }\vec{\nabla}\times\vec{B}_\gamma=-m_\gamma^2(\ \vec{A}^{(Dirac)}+\vec{A}_\gamma). \tag{7}$$

The vector $\vec{A}^{(Dirac)}$ is the standard singular vector potential representing the field of a fixed monopole,

$$\vec{A}^{(Dirac)}(\vec{r})=\frac{e_m}{r^2}\frac{\sin(\theta)}{1+\cos(\theta)}(\hat{n}\times\vec{r}). \tag{8}$$

where e_m is the magnetic charge. The vector potential $\vec{A}_\gamma(\vec{r})$ is given by the following general expression,

$$\vec{A}_\gamma(\vec{r})=e_m m_\gamma^2 f_\gamma(m_\gamma r,m_\gamma\vec{r}\cdot\hat{n})(\hat{n}\times\vec{r}) \tag{9}$$

where f_γ is a generic scalar field function. Because of the second equation in (7), it is clear that no spherically symmetric diffuse magnetic field solutions are allowed, that is to say, solutions like

$$\vec{B}_\gamma(\vec{r})=B_\gamma(r)\hat{r} \tag{10}$$

are not allowed.

On the other hand, it is well known that the classical non-relativistic theory describing the standard electromagnetic scattering of an electric charge from a magnetic monopole does have a Hamiltonian formulation since there is a conserved total angular momentum [7]. With this result in mind, let us consider the classical non-relativistic theory describing a point-like electric particle with charge e and mass m moving in the field of a fixed monopole of charge e_m, but let us suppose that the electromagnetic interaction is mediated by massive photons. The classical non-relativistic spherically symmetric Hamiltonian describing this physical system can be written as

$$H(\vec{p},\vec{r})=\frac{(\vec{p}\cdot\vec{r})^2}{2mr^2}+\frac{(\vec{J}^2-\vec{s}^2)}{2mr^2}. \tag{11}$$

The vector $\vec{J}$ is the conserved total angular momentum, the generator of spatial rotations, as will be explicitly shown below,

$$\vec{J}=\vec{r}\times\left[\vec{p}-\frac{e}{c}(\vec{A}_\gamma+\vec{A}^{(Dirac)})\right]+\vec{s} \tag{12}$$

while $\vec{s}$ is the angular momentum of the electromagnetic field,

$$\vec{s} \stackrel{\text{def}}{=} \vec{L}_{EM-field} = (8\pi c)^{-1} \int \left[\vec{r} \times (\vec{E} \times \vec{B}) \right] d^3\vec{r}. \tag{13}$$

The vector $\vec{s}$ is also known as the Poincaré magnetic angular momentum and it may be interpreted as a "classical spin". As such, $\vec{s}$ is taken as an angular momentum with independent degrees of freedom obeying the following classical Poisson-bracket relation,

$$\{s_i,\, s_j\} = -\varepsilon_{ijk} s_k. \tag{14}$$

The above Poisson brackets between two generic functions $f(\vec{p},\, \vec{r},\, t)$ and $g(\vec{p},\, \vec{r},\, t)$ of the dynamical variables $\vec{p}$ and $\vec{r}$, are defined as,

$$\{f(\vec{p},\, \vec{r},\, t),\, g(\vec{p},\, \vec{r},\, t)\} \stackrel{\text{def}}{=} \sum_i (\partial_{p_i} f \partial_{r_i} g - \partial_{r_i} f \partial_{p_i} g) \tag{15}$$

and the basic canonical Poisson bracket structure for the conjugate variables is given by,

$$\{r_i,\, r_j\} = 0,\ \{r_i,\, p_j\} = -\delta_{ij},\ \{p_i,\, p_j\} = 0. \tag{16}$$

Let us show explicitly that $\vec{J}$ is the generator of spatial rotations so that we can safely define the rank of a tensor by studying its transformation rules under such rotations. Let us prove,

$$\{J_i,\, J_j\} = -\varepsilon_{ijk} J_k. \tag{17}$$

Using the tensorial notation for the cross product appearing in the definition of $\vec{J}$, and using the standard properties of a well-define Poisson bracket structure, the brackets in equation (17) become,

$$\begin{aligned} \{J_i,\, J_l\} \;=\; & \{\varepsilon_{ijk} r_j p_k,\, \varepsilon_{lmn} r_m p_n\} - \{\varepsilon_{ijk} r_j p_k,\, \varepsilon_{lmn} r_m A_n\} + \\ & - \{\varepsilon_{ijk} r_j A_k,\, \varepsilon_{lmn} r_m p_n\} + \{\varepsilon_{ijk} r_j A_k,\, \varepsilon_{lmn} r_m A_n\} + \{s_i,\, s_l\}\,. \end{aligned} \tag{18}$$

Using the basic canonical Poisson bracket structure expressed in (16) and the standard properties of Poisson brackets together with the following identity,

$$\varepsilon_{ijk}\varepsilon_{mlk} = \delta_{im}\delta_{jl} - \delta_{il}\delta_{jm} \tag{19}$$

the first bracket on the *rhs* of (18) becomes,

$$\{\varepsilon_{ijk} r_j p_k,\, \varepsilon_{lmn} r_m p_n\} = r_l p_i - r_i p_l. \tag{20}$$

Similarly, the second, the third and the fourth brackets on the *rhs* of (18) become,

$$-\{\varepsilon_{ijk} r_j p_k,\, \varepsilon_{lmn} r_m A_n\} = \delta_{il} r_n A_n - r_l A_i + \varepsilon_{ijk}\varepsilon_{lmn} r_m p_k \{A_n,\, r_j\} \tag{21}$$

$$-\{\varepsilon_{ijk} r_j A_k,\, \varepsilon_{lmn} r_m p_n\} = -\delta_{il} r_k A_k + r_i A_l + \varepsilon_{ijk}\varepsilon_{lmn} r_j p_n \{r_m,\, A_k\} \tag{22}$$

$$\{\varepsilon_{ijk} r_j A_k,\, \varepsilon_{lmn} r_m A_n\} = -\varepsilon_{ijk}\varepsilon_{lmn} r_j A_n \{r_m,\, A_k\} - \varepsilon_{ijk}\varepsilon_{lmn} r_m A_k \{A_n,\, r_j\}\,. \tag{23}$$

The last bracket on the *rhs* of (18) is given by (14). Finally, substituting these five brackets in the *rhs* of (18) and ordering them properly, the Poisson brackets of $\vec{J}$ become,

$$\begin{aligned} \{J_i,\, J_l\} \;=\; & (r_l p_i - r_i p_l - r_l A_i + r_i A_l - \varepsilon_{ilm} s_m) \;+ \\ & +\varepsilon_{ijk}\varepsilon_{lmn}\left[r_m p_k \{A_n,\, r_j\} - r_j p_n \{r_m,\, A_k\}\right] + \\ & +\varepsilon_{ijk}\varepsilon_{lmn}\left[r_j A_n \{A_k,\, r_m\} - r_m A_k \{A_n,\, r_j\}\right]. \end{aligned} \tag{24}$$

Because of the full antisymmetry of the Levi-Civita tensor,

$$\begin{aligned} & \varepsilon_{ijk}\varepsilon_{lmn} r_m p_k \{A_n,\, r_j\} - \varepsilon_{ijk}\varepsilon_{lmn} r_j p_n \{A_n,\, r_m\} \;= \\ & \qquad (\varepsilon_{ijk}\varepsilon_{lmn} - \varepsilon_{imn}\varepsilon_{ljk})\, r_m p_k \{A_n,\, r_j\} \;=\; 0. \end{aligned} \tag{25}$$

Therefore, equation (24) becomes,

$$\begin{aligned} \{J_i,\, J_l\} &= r_l p_i - r_i p_l - r_l A_i + r_i A_l - \varepsilon_{ilm} s_m = \\ &= -\varepsilon_{ilm}\left[\varepsilon_{mnk} r_n (p_k - A_k) + s_m\right]. \end{aligned} \tag{26}$$

Using equation (19), we obtain

$$-\varepsilon_{ilm}\varepsilon_{mnk} r_n p_k = (r_l p_i - r_i p_l), \text{ and } \ \varepsilon_{ilm}\varepsilon_{mnk} r_n A_k = -(r_l A_i - r_i A_l) \tag{27}$$

and finally,

$$\{J_i,\, J_l\} = -\varepsilon_{ilm} J_m. \tag{28}$$

This concludes our proof: $\vec{J}$ defined in (12) represents the conserved total angular momentum of the system and it is the generator of spatial rotations. For further details of this proof, see the appendix.

At this point, we have all the elements to show the classical inconsistency of the problem. Let us define the generalized momentum vector as,

$$\vec{P} \stackrel{\text{def}}{=} \vec{p} - \frac{e}{c}\vec{A}, \qquad \vec{A} = \vec{A}_\gamma + \vec{A}^{(Dirac)}. \tag{29}$$

Let us assume that there exist a well-defined Poisson bracket structure in the classical theoretical setting in consideration. In particular, let us assume a well-defined classical Poisson bracket structure among the vector fields $\vec{J}$, $\vec{P}$, and $\vec{r}$, that is,

$$\{J_i,\, J_j\} = -\varepsilon_{ijk} J_k,\ \{J_i,\, r_j\} = -\varepsilon_{ijk} r_k,\ \{J_i,\, P_j\} = -\varepsilon_{ijk} P_k. \tag{30}$$

Being $\vec{J}$ the generator of rotations, it is required that any arbitrary vector $\vec{v}$ must satisfy the following classical commutation rules,

$$\{J_i,\, v_j\} = -\varepsilon_{ijk} v_k. \tag{31}$$

Therefore, let us study the transformation properties of the magnetic field under spatial rotations. It must be,

$$\{J_i,\, B_j\} = -\varepsilon_{ijk} B_k. \tag{32}$$

In terms of the magnetic field decomposition, equation (32) is equivalent to,

$$\left\{J_i,\, B_j^{(Dirac)}\right\} = -\varepsilon_{ijk} B_k^{(Dirac)} \tag{33}$$

and,

$$\{J_i,\, (B_\gamma)_j\} = -\varepsilon_{ijk} (B_\gamma)_k. \tag{34}$$

It is quite straightforward to check the validity of (33), as a matter of fact,

$$\begin{aligned} \left\{J_i,\, B_j^{(Dirac)}\right\} &= \left\{J_i,\, \frac{e_m}{r^3} r_j\right\} = \frac{e_m}{r^3}\{J_i,\, r_j\} + \left\{J_i,\, \frac{e_m}{r^3}\right\} r_j \\ &= -\varepsilon_{ijk}\frac{e_m}{r^3} r_k \equiv -\varepsilon_{ijk} B_k^{(Dirac)}. \end{aligned} \tag{35}$$

Let us consider the validity of equation (32), where, in terms of the total vector potential $\vec{A}$, the total magnetic field is

$$B_j = \varepsilon_{jlm}\partial_l A_m. \tag{36}$$

Fixing the constants c and e equal to one for the sake of convenience, let us consider first the Poisson brackets of the generalized momentum vector components. Using (16), the standard properties of Poisson brackets together with equations (19) and (36), we obtain,

$$\{P_i,\, P_j\} = -\varepsilon_{ijk} B_k. \tag{37}$$

Multiplying both sides of (37) by ε_{ijn}, we obtain

$$\varepsilon_{ijn}\left\{P_i,\, P_j\right\} = -\varepsilon_{ijn}\varepsilon_{ijk}B_k = -2\delta_{nk}B_k = -2B_n \tag{38}$$

and therefore,

$$B_k = -\frac{1}{2}\varepsilon_{ijk}\left\{P_i,\, P_j\right\}. \tag{39}$$

Therefore, substituting B_k of equation (39) into (32), we obtain

$$\left\{J_i,\, B_j\right\} = -\frac{1}{2}\varepsilon_{lmj}\left\{J_i,\, \left\{P_l,\, P_m\right\}\right\}. \tag{40}$$

The double commutator in equation (40) cannot be calculated in a direct way. However, because we are assuming the existence of a well-defined Poisson bracket structure among the vectors $\vec{J}$, $\vec{B}$ and $\vec{r}$, this double commutator can be evaluated by using the following Jacobi identity,

$$\left\{J_i,\, \left\{P_l,\, P_m\right\}\right\} + \left\{P_m,\, \left\{J_i,\, P_l\right\}\right\} + \left\{P_l,\, \left\{P_m,\, J_i\right\}\right\} = 0. \tag{41}$$

Thus, using the fact that $\vec{J}$ is the generator of rotations, that $\vec{P}$ transforms as a vector quantity under rotations, and using equation (19), we obtain

$$\left\{J_i,\, \left\{P_l,\, P_m\right\}\right\} = -\delta_{il}B_m + \delta_{im}B_l. \tag{42}$$

Substituting equations (39) into (42), we obtain

$$\left\{J_i,\, B_j\right\} = -\varepsilon_{ijm}B_m. \tag{43}$$

Therefore, we have shown that in a pure classical theoretical framework given by the Poisson brackets formalism, the commutation rule between the generator of spatial rotations and the total magnetic field is expressed in (43). Our last step is to calculate the Poisson brackets between $\vec{J}$ and the magnetic field $\vec{B}_\gamma$. Using equation (6), standard Poisson brackets properties and the fact that $\vec{J}$ is the generator of rotations, these brackets become,

$$\left\{J_i,\, (\vec{B}_\gamma)_j\right\}_{Poisson} = -\varepsilon_{ijk}(B_\gamma)_k + \left\{J_i,\, b_\gamma^{(1)}\right\} r_j + \left\{J_i,\, b_\gamma^{(2)}\right\} n_j. \tag{44}$$

In order to have proper Poisson brackets, for each vectors $\hat{n}$ and $\vec{r}$, the following relation must hold

$$\left\{J_i,\, b_\gamma^{(1)}\right\} r_j + \left\{J_i,\, b_\gamma^{(2)}\right\} n_j = 0. \tag{45}$$

Observe that the second Poisson bracket in the *rhs* of (44) contains a term quadratic in n_k,

$$\begin{aligned}
\left\{J_i,\, b_\gamma^{(2)}\right\} n_j &= (\partial_{p_k}J_i)(\partial_{r_k}b_\gamma^{(2)})n_j = (\partial_{p_k}J_i)\left[\partial_r b_\gamma^{(2)}\frac{r_k}{r} + \partial_{\left(\vec{r}\cdot\hat{n}\right)}b_\gamma^{(2)}n_k\right]n_j \\
&= \frac{1}{r}\partial_{p_k}J_i\;\partial_r b_\gamma^{(2)}r_k n_j + \partial_{p_k}J_i\partial_{\left(\vec{r}\cdot\hat{n}\right)}b_\gamma^{(2)}n_k n_j.
\end{aligned} \tag{46}$$

Since, the proper Poisson brackets should be linear in n_k, we require

$$\partial_{\left(\vec{r}\cdot\hat{n}\right)}b_\gamma^{(2)} = 0. \tag{47}$$

There is no way to cancel out this term in (44), then it must be,

$$b_\gamma^{(2)} = 0. \tag{48}$$

We now consider the first Poisson bracket on the *rhs* of (44). Because of the anti-symmetry in the indices i and j of the term $\varepsilon_{ijk}(B_\gamma)_k$, it must be

$$\left\{J_i,\, b_\gamma^{(1)}\right\} r_j + \left\{J_j,\, b_\gamma^{(1)}\right\} r_i = 0 \tag{49}$$

that is,

$$\left\{J_i,\, b_\gamma^{(1)}\right\} r_i = 0. \tag{50}$$

Explicitly, equation (50) becomes,

$$\begin{aligned} 0 &= (\partial_{p_k} J_i)(\partial_{r_k} b_\gamma^{(1)}) r_i = (\partial_{p_k} J_i)\left[\partial_r b_\gamma^{(1)} \frac{r_k}{r} + \partial_{\left(\vec{r}\cdot\hat{n}\right)} b_\gamma^{(1)} n_k\right] r_i = \\ &= \frac{1}{r}(\partial_{p_k} J_i)(\,\partial_r b_\gamma^{(1)}) r_k r_i + (\partial_{p_k} J_i)(\partial_{\left(\vec{r}\cdot\hat{n}\right)} b_\gamma^{(1)}) n_k r_i. \end{aligned} \tag{51}$$

We neglect the quadratic term in r_k in equation (51) since this term has no analog in the proper Poisson brackets. Then, we have

$$\partial_{\left(\vec{r}\cdot\hat{n}\right)} b_\gamma^{(1)} = 0. \tag{52}$$

Recalling that

$$\hat{n} = -\hat{z} = -\left\{\cos(\theta)\hat{r} - \sin(\theta)\hat{\theta}\right\} = -\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta} \tag{53}$$

then,

$$\hat{n}\cdot\hat{r} = -\cos(\theta) = \theta - \text{dependent}. \tag{54}$$

Therefore, equation (52) is satisfied by an arbitrary scalar function $b_\gamma(r)$. As a consequence, the magnetic field $\vec{B}_\gamma$ is not $\theta-$dependent (in a more general situation in which $\hat{n}$ is not along the z-axis, we would conclude that the magnetic field is not $(\theta,\varphi)-$dependent). $\vec{B}_\gamma$ must be a spherically symmetric field whose general expression is the following,

$$\vec{B}_\gamma\left(\vec{r}\right) = B_\gamma(r)\hat{r}. \tag{55}$$

In conclusion, in order to have a well-defined classical Poisson bracket structure in the problem under investigation, one must deal with diffuse magnetic field solutions exhibiting spherical symmetry. However, those very same solutions are not compatible with massive classical electrodynamics with magnetic monopoles. This result means that it is not possible to formulate a consistent non-relativistic classical theory describing the finite-range electromagnetic interaction between a point-like electric charge and a fixed Dirac monopole without a string. In other words, there is no way to construct a consistent Lie algebra in our classical framework and this leads to the conclusion that there is no angular momentum to be quantized in order to give the Dirac quantization rule. This fact points out that the string attached to the monopole is visible and there is no way to make it invisible when considering finite-range electromagnetic interactions in a pure classical framework. An important feature of our approach is that we do not use any kind of semiclassical approximation or limiting procedure for $\hbar \to 0$.

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APPENDIX A: THE GENERATOR OF SPATIAL ROTATIONS

We show that $\vec{J}$ is the generator of spatial rotations, that is,

$$\{J_i,\, J_j\} = -\varepsilon_{ijk} J_k. \tag{A1}$$

Notice that,

$$\begin{aligned} \{J_i,\, J_l\} &= \{\varepsilon_{ijk} r_j \left(p_k - A_k\right) + s_i,\, \varepsilon_{lmn} r_m \left(p_n - A_n\right) + s_l\} \\ &= \{\varepsilon_{ijk} r_j p_k - \varepsilon_{ijk} r_j A_k + s_i,\, \varepsilon_{lmn} r_m p_n - \varepsilon_{lmn} r_m A_n + s_l\} \\ &= \{\varepsilon_{ijk} r_j p_k,\, \varepsilon_{lmn} r_m p_n\} - \{\varepsilon_{ijk} r_j p_k,\, \varepsilon_{lmn} r_m A_n\} - \{\varepsilon_{ijk} r_j A_k,\, \varepsilon_{lmn} r_m p_n\} + \\ &\quad + \{\varepsilon_{ijk} r_j A_k,\, \varepsilon_{lmn} r_m A_n\} + \{s_i,\, s_l\}\,. \end{aligned} \tag{A2}$$

Therefore there are five Poisson brackets to be calculated. Consider the first one,

$$\begin{aligned}
\{\varepsilon_{ijk}r_jp_k,\, \varepsilon_{lmn}r_mp_n\} &= \varepsilon_{ijk}\varepsilon_{lmn}\{r_jp_k,\, r_mp_n\} = \varepsilon_{ijk}\varepsilon_{lmn}\left[r_j\{p_k,\, r_mp_n\} + \{r_j,\, r_mp_n\}p_k\right] \\
&= \varepsilon_{ijk}\varepsilon_{lmn}\left[-r_j\{r_mp_n,\, p_k\} - \{r_mp_n,\, r_j\}p_k\right] \\
&= \varepsilon_{ijk}\varepsilon_{lmn}\left[-r_j\left(r_m\{p_n,\, p_k\} + \{r_m,\, p_k\}p_n\right)\right] + \varepsilon_{ijk}\varepsilon_{lmn}\left[-\left(r_m\{p_n,\, r_j\} + \{r_m,\, r_j\}p_n\right)p_k\right] \\
&= \varepsilon_{ijk}\varepsilon_{lmn}\left[\delta_{mk}r_jp_n - \delta_{nj}r_mp_k\right] = \varepsilon_{ijk}\varepsilon_{lmn}\delta_{mk}r_jp_n - \varepsilon_{ijk}\varepsilon_{lmn}\delta_{nj}r_mp_k \\
&= \varepsilon_{ijk}\varepsilon_{lkn}r_jp_n - \varepsilon_{ink}\varepsilon_{lmn}r_mp_k = -\varepsilon_{ijk}\varepsilon_{lnk}r_jp_n + \varepsilon_{ikn}\varepsilon_{lmn}r_mp_k \\
&= -\left(\delta_{il}\delta_{jn} - \delta_{in}\delta_{jl}\right)r_jp_n + \left(\delta_{il}\delta_{km} - \delta_{im}\delta_{lk}\right)r_mp_k \\
&= -\delta_{il}\delta_{jn}r_jp_n + \delta_{in}\delta_{jl}r_jp_n + \delta_{il}\delta_{km}r_mp_k - \delta_{im}\delta_{lk}r_mp_k \\
&= -\delta_{il}r_np_n + r_lp_i + \delta_{il}r_kp_k - r_ip_l = r_lp_i - r_ip_l
\end{aligned} \tag{A3}$$

thus,

$$\{\varepsilon_{ijk}r_jp_k,\, \varepsilon_{lmn}r_mp_n\} = r_lp_i - r_ip_l. \tag{A4}$$

Consider the second bracket,

$$\begin{aligned}
-\{\varepsilon_{ijk}r_jp_k,\, \varepsilon_{lmn}r_mA_n\} &= -\varepsilon_{ijk}\varepsilon_{lmn}\{r_jp_k,\, r_mA_n\} \\
&= -\varepsilon_{ijk}\varepsilon_{lmn}\left[r_j\{p_k,\, r_mA_n\} + \{r_j,\, r_mA_n\}p_k\right] \\
&= -\varepsilon_{ijk}\varepsilon_{lmn}\left[-r_j\{r_mA_n,\, p_k\} - \{r_mA_n,\, r_j\}p_k\right] \\
&= -\varepsilon_{ijk}\varepsilon_{lmn}\left[-r_jr_m\{A_n,\, p_k\} - r_j\{r_m,\, p_k\}A_n\right] + \\
&\quad -\varepsilon_{ijk}\varepsilon_{lmn}\left[-r_m\{A_n,\, r_j\}p_k - \{r_m,\, r_j\}A_np_k\right] \\
&= -\varepsilon_{ijk}\varepsilon_{lmn}\left[\delta_{mk}r_jA_n - r_mp_k\{A_n,\, r_j\}\right] \\
&= -\varepsilon_{ijk}\varepsilon_{lkn}r_jA_n + \varepsilon_{ijk}\varepsilon_{lmn}r_mp_k\{A_n,\, r_j\} \\
&= \varepsilon_{ijk}\varepsilon_{l,n,k}r_jA_n + \varepsilon_{ijk}\varepsilon_{lmn}r_mp_k\{A_n,\, r_j\} \\
&= \left(\delta_{il}\delta_{jn} - \delta_{in}\delta_{jl}\right)r_jA_n + \varepsilon_{ijk}\varepsilon_{lmn}r_mp_k\{A_n,\, r_j\} \\
&= \delta_{il}\delta_{jn}r_jA_n - \delta_{in}\delta_{jl}r_jA_n + \varepsilon_{ijk}\varepsilon_{lmn}r_mp_k\{A_n,\, r_j\} \\
&= \delta_{il}r_nA_n - r_lA_i + \varepsilon_{ijk}\varepsilon_{lmn}r_mp_k\{A_n,\, r_j\}
\end{aligned} \tag{A5}$$

thus,

$$-\{\varepsilon_{ijk}r_jp_k,\, \varepsilon_{lmn}r_mA_n\} = \delta_{il}r_nA_n - r_lA_i + \varepsilon_{ijk}\varepsilon_{lmn}r_mp_k\{A_n,\, r_j\}. \tag{A6}$$

Using the standard canonical algebra, the third bracket becomes,

$$-\{\varepsilon_{ijk}r_jA_k,\, \varepsilon_{lmn}r_mp_n\} = -\delta_{il}r_kA_k + r_iA_l + \varepsilon_{ijk}\varepsilon_{lmn}r_jp_n\{r_m,\, A_k\}. \tag{A7}$$

For the fourth bracket, we obtain

$$\begin{aligned}
\{\varepsilon_{ijk}r_jA_k,\, \varepsilon_{lmn}r_mA_n\} &= \varepsilon_{ijk}\varepsilon_{lmn}\{r_jA_k,\, r_mA_n\} \\
&= \varepsilon_{ijk}\varepsilon_{lmn}\left[r_j\{A_k,\, r_mA_n\} + \{r_j,\, r_mA_n\}A_k\right] \\
&= \varepsilon_{ijk}\varepsilon_{lmn}\left[-r_j\{r_mA_n,\, A_k\} - \{r_mA_n,\, r_j\}A_k\right] \\
&= \varepsilon_{ijk}\varepsilon_{lmn}\left[-r_j\{r_m,\, A_k\}A_n - r_m\{A_n,\, r_j\}A_k\right] \\
&= -\varepsilon_{ijk}\varepsilon_{lmn}r_jA_n\{r_m,\, A_k\} - \varepsilon_{ijk}\varepsilon_{lmn}r_mA_k\{A_n,\, r_j\}.
\end{aligned} \tag{A8}$$

For the last bracket, let us remind that the vector s is such the Poisson brackets of its components satisfy equation (14). In conclusion, using equations $(A4)$, $(A6)$, $(A7)$, $(A8)$ and using the commutation rules of the classical spin, equation $(A2)$ becomes,

$$\begin{aligned}
\{J_i,\, J_l\} &= r_lp_i - r_ip_l + \delta_{il}r_nA_n - r_lA_i + \varepsilon_{ijk}\varepsilon_{lmn}r_mp_k\{A_n,\, r_j\} - \delta_{il}r_kA_k + \\
&\quad + r_iA_l + \varepsilon_{ijk}\varepsilon_{lmn}r_jp_n\{r_m,\, A_k\} - \varepsilon_{ijk}\varepsilon_{lmn}r_jA_n\{r_m,\, A_k\} + \\
&\quad -\varepsilon_{ijk}\varepsilon_{lmn}r_mA_k\{A_n,\, r_j\} - \varepsilon_{ilm}s_m \\
&= \left(r_lp_i - r_ip_l - r_lA_i + r_iA_l - \varepsilon_{ilm}s_m\right) + \\
&\quad + (\varepsilon_{ijk}\varepsilon_{lmn}\left[r_mp_k\{A_n,\, r_j\} - r_jp_n\{r_m,\, A_k\} + r_jA_n\{A_k,\, r_m\} - r_mA_k\{A_n,\, r_j\}\right]).
\end{aligned} \tag{A9}$$

Notice that,

$$\varepsilon_{ijk}\varepsilon_{lmn}r_m p_k \{A_n,\, r_j\} - \varepsilon_{ijk}\varepsilon_{lmn}r_j p_n \{A_n,\, r_m\} = (\varepsilon_{ijk}\varepsilon_{lmn} - \varepsilon_{imn}\varepsilon_{ljk})\, r_m p_k \{A_n,\, r_j\} = 0. \tag{A10}$$

If $i = l$, then,

$$\varepsilon_{ijk}\varepsilon_{lmn} - \varepsilon_{imn}\varepsilon_{ljk} = \varepsilon_{ijk}\varepsilon_{imn} - \varepsilon_{imn}\varepsilon_{ijk} \equiv 0. \tag{A11}$$

If $i \neq l$, let us say $i = 1$ and $l = 2$, then

$$\varepsilon_{ijk}\varepsilon_{lmn} - \varepsilon_{imn}\varepsilon_{ljk} = \varepsilon_{1jk}\varepsilon_{2mn} - \varepsilon_{1mn}\varepsilon_{2jk}. \tag{A12}$$

Therefore, the possible non-vanishing pieces are:

$$\varepsilon_{123}\varepsilon_{213} - \varepsilon_{213}\varepsilon_{123} \equiv 0,\ \varepsilon_{132}\varepsilon_{231} - \varepsilon_{231}\varepsilon_{132} \equiv 0,\ \varepsilon_{132}\varepsilon_{213} - \varepsilon_{231}\varepsilon_{123} \equiv 0, \text{ etc. etc.} \tag{A13}$$

Therefore, equation $(A9)$ becomes,

$$\begin{aligned} \{J_i,\, J_l\} &= (r_l p_i - r_i p_l - r_l A_i + r_i A_l - \varepsilon_{ilm} s_m) \\ &= -\varepsilon_{ilm}\left[\varepsilon_{mnk} r_n (p_k - A_k) + s_m\right] \\ &= -\varepsilon_{ilm} J_m. \end{aligned} \tag{A14}$$

Indeed,

$$\begin{aligned} -\varepsilon_{ilm}\varepsilon_{mnk} r_n p_k &= -\varepsilon_{ilm}\varepsilon_{kmn} r_n p_k = -\varepsilon_{ilm}\varepsilon_{nkm} r_n p_k = \\ &= (\delta_{in}\delta_{lk} - \delta_{ik}\delta_{l,n})\, r_n p_k = -\delta_{in}\delta_{lk} r_n p_k + \delta_{ik}\delta_{l,n} r_n p_k \\ &= -r_i p_l + r_l p_i = (r_l p_i - r_i p_l) \end{aligned} \tag{A15}$$

and,

$$\varepsilon_{ilm}\varepsilon_{mnk} r_n A_k = r_i A_l + r_l A_i = -(r_l A_i - r_i A_l). \tag{A16}$$

This concludes our proof.

APPENDIX B: THE JACOBI IDENTITY

Consider the generalized momentum vector,

$$\vec{P} \overset{\text{def}}{=} \vec{p} - \frac{e}{c}\vec{A},\ \vec{A} = \vec{A}_\gamma + \vec{A}^{(Dirac)}. \tag{B1}$$

Consider the Poisson bracket of the generalized momentum vector components,

$$\begin{aligned} \{P_i,\, P_j\} &= \{p_i - A_i,\, p_j - A_j\} = \\ &= \{p_i,\, p_j\} - \{p_i,\, A_j\} - \{A_i,\, p_j\} + \{A_i,\, A_j\} = \{A_j,\, p_i\} - \{A_i,\, p_j\} \\ &= \{A_j,\, p_i\} - \{A_i,\, p_j\} = -\partial i A_j + \partial_j A_i = -(\partial i A_j - \partial_j A_i) \\ &= -\varepsilon_{ijk} B_k \end{aligned} \tag{B2}$$

where

$$B_j = \varepsilon_{jlm}\partial_l A_m. \tag{B3}$$

Using the fact that $\{J_i, B_j\} = -\varepsilon_{ijk} B_k$ and the identity $\varepsilon_{ijk}\varepsilon_{mlk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, it follows that,

$$\begin{aligned} \varepsilon_{ijk} B_k &= \varepsilon_{ijk}\varepsilon_{klm}\partial_l A_m = \varepsilon_{ijk}\varepsilon_{mkl}\partial_l A_m = -\varepsilon_{ijk}\varepsilon_{mlk}\partial_l A_m \\ &= -(\delta_{im}\delta_{jl} - \delta_{il}\delta_{jm})\partial_l A_m = -\delta_{im}\delta_{jl}\partial_l A_m + \delta_{il}\delta_{jm}\partial_l A_m \\ &= -\delta_{im}\partial_j A_m + \delta_{il}\partial_l A_j = \partial_i A_j - \partial_j A_i. \end{aligned} \tag{B4}$$

Using equation $(B2)$, we obtain

$$\varepsilon_{ijn}\{P_i,\, P_j\}_{Poisson} = -\varepsilon_{ijn}\varepsilon_{ijk} B_k = -2\delta_{nk} B_k = -2B_n. \tag{B5}$$

Thus,

$$B_k = -\frac{1}{2}\varepsilon_{ijk}\left\{P_i,\, P_j\right\}. \tag{B6}$$

Finally, let us focus on the following Poisson bracket,

$$\{J_i,\, B_j\} = \left\{J_i, -\frac{1}{2}\varepsilon_{lmj}\left\{P_l,\, P_m\right\}\right\} = -\frac{1}{2}\varepsilon_{lmj}\left\{J_i,\, \{P_l,\, P_m\}\right\}. \tag{B7}$$

Using the Jacobi identity,

$$\{J_i,\, \{P_l,\, P_m\}\} + \{P_m,\, \{J_i,\, P_l\}\} + \{P_l,\, \{P_m,\, J_i\}\} = 0 \tag{B8}$$

we obtain

$$\begin{aligned}
\{J_i,\, \{P_l,\, P_m\}\} &= -\{P_m,\, \{J_i,\, P_l\}\} - \{P_l,\, \{P_m,\, J_i\}\} = \{P_l,\, \{J_i,\, P_m\}\} - \{P_m,\, \{J_i,\, P_l\}\} \\
&= \{P_l,\, -\varepsilon_{imk}P_k\} - \{P_m,\, -\varepsilon_{ilk}P_k\} = -\varepsilon_{imk}\{P_l,\, P_k\} + \varepsilon_{ilk}\{P_m,\, P_k\} \\
&= -\varepsilon_{imk}(-\varepsilon_{lkq}B_q) + \varepsilon_{ilk}(-\varepsilon_{mkq}B_q) = \varepsilon_{imk}\varepsilon_{lkq}B_q - \varepsilon_{ilk}\varepsilon_{mkq}B_q \\
&= -\varepsilon_{imk}\varepsilon_{lqk}B_q + \varepsilon_{ilk}\varepsilon_{mqk}B_q = -(\delta_{il}\delta_{mq} - \delta_{iq}\delta_{ml})B_q + (\delta_{im}\delta_{lq} - \delta_{iq}\delta_{lm})B_q \\
&= -\delta_{il}\delta_{mq}B_q + \delta_{iq}\delta_{ml}B_q + \delta_{im}\delta_{lq}B_q - \delta_{iq}\delta_{lm}B_q \\
&= -\delta_{il}B_m + \delta_{ml}B_i + \delta_{im}B_l - \delta_{lm}B_i = -\delta_{il}B_m + \delta_{im}B_l.
\end{aligned} \tag{B9}$$

Then, using equations ($B6$) and ($B9$), we obtain

$$\begin{aligned}
\{J_i,\, B_j\} &= -\frac{1}{2}\varepsilon_{lmj}(-\delta_{il}B_m + \delta_{im}B_l) = \frac{1}{2}\varepsilon_{lmj}\delta_{il}B_m - \frac{1}{2}\varepsilon_{lmj}\delta_{im}B_l \\
&= \frac{1}{2}\varepsilon_{imj}B_m - \frac{1}{2}\varepsilon_{lij}B_l = -\frac{1}{2}\varepsilon_{ijm}B_m - \frac{1}{2}\varepsilon_{mij}B_m \\
&= -\frac{1}{2}\varepsilon_{ijm}B_m - \frac{1}{2}\varepsilon_{ijm}B_m = -\varepsilon_{ijm}B_m.
\end{aligned} \tag{B10}$$

We have shown that in a pure classical theoretical framework given by the Poisson brackets formalism, the commutation rule between the generator of spatial rotations and the total magnetic field is,

$$\{J_i, B_j\} = i\varepsilon_{ijk}B_k. \tag{B11}$$

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[1] P. A. M. Dirac, "Quantized singularities in the electromagnetic field", Proc. Roy. Soc. **A133**, 60 (1931); P. A. M. Dirac, "The Theory of Magnetic Poles", Phys. Rev. **74**, 817 (1948).
[2] R. Jackiw, "Three-Cocycle in Mathematics and Physics", Phys. Rev. Lett. **54**, 159 (1985).
[3] G. t'Hooft, "Magnetic monopoles in unified gauge theories", Nucl. Phys. **B79**, 276 (1974).
[4] A. Polyakov, "Particle spectrum in quantum field theory", JEPT Lett. **20**, 194 (1974).
[5] M. Israelit, "Massive Electrodynamics and Magnetic Monopoles", Gen. Rel. Grav. **29** (1997) 1411-1424.
[6] M. Israelit, "Magnetic Monopoles and Massive Photons in a Weyl-type Electrodynamics", Gen. Rel. Grav. **29** (1997) 1597-1614.
[7] A.S. Goldhaber, "Role of Spin in the Monopole Problem", Phys. Rev. D **140**, B1407 (1965).
[8] A. Yu Ignatiev and G. C. Joshi, "Massive Electrodynamics and the Magnetic Monopoles", Phys. Rev. D **53**, 984 (1995).